A NOTE ON ESTIMATION OF THE GLOBAL INTENSITY OF A CYCLIC POISSON PROCESS IN THE PRESENCE OF LINEAR TREND

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Abstract. We construct and investigate a consistent kernel-type nonparametric estimator of the global intensity of a cyclic Poisson process in the presence of linear trend. It is assumed that only a single realization of the Poisson process is observed in a bounded window. We prove that the proposed estimator is consistent when the size of the window indefinitely expands. The asymptotic bias and variance of the proposed estimator are computed. Bias reduction of the estimator is also proposed.

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1. Introduction

Let X be a Poisson point process on $[0, \infty)$ with absolutely continuous σ-finite mean measure μ w.r.t. Lebesgue measure ν and with (unknown) locally integrable intensity function λ, i.e., for any bounded Borel set B we have

$$\mu(B) = \mathbf{E}N(B) = \int_B \lambda(s)ds < \infty.$$

Furthermore, λ is assumed to consist of two components, namely a periodic or cyclic component with period $\tau > 0$ and a (unknown) linear trend component. In other words, for any point $s \in [0, \infty)$, we can write the intensity function λ as

$$\lambda(s) = \lambda_c(s) + as \tag{1.1}$$

where $\lambda_c(s)$ is a periodic function with period τ and a denotes the slope of the linear trend. In the present paper, we do not assume any (parametric) form of λ_c except that it is periodic. That is we assume that the equality

$$\lambda_c(s + k\tau) = \lambda_c(s) \tag{1.2}$$

holds for all $s \in [0, \infty)$ and $k \in \mathbf{Z}$. Here we consider a Poisson point process on $[0, \infty)$ instead of, for instance, on $\mathbf{R}$ because λ has to satisfy (1.1) and must be non negative. For the same reason we also restrict our attention to the case $a > 0$.

Furthermore, let $W_1,\ W_2, \ldots$ be a sequence of intervals $[0, |W_n|]$, $n = 1, 2, \ldots$, such that the size or the Lebesgue measure $\nu(W_n) = |W_n|$ of W_n is finite for each fixed $n \in \mathbf{N}$, but

$$|W_n| \to \infty, \tag{1.3}$$

as $n \to \infty$.

Suppose now that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ (cf. (1.1)) is observed, though only within a bounded interval, called 'window' $W \subset [0, \infty)$. Our goal in this paper is to construct a consistent non-parametric estimator of the global intensity

$$\theta = \frac{1}{\tau}\mu([0, \tau]) = \frac{1}{\tau}\int_0^\tau \lambda_c(s)ds \tag{1.4}$$

of the cyclic component λ_c of λ in (1.1), using only a single realization $X(\omega)$ of the Poisson process X observed in $W := W_n$. We also compute the asymptotic bias and variance of the proposed estimator. The present paper aims at extending previous work for the purely cyclic case, i.e. $a = 0$, (cf. [2]) to the more general model (1.1).

Parallel to this paper, Helmers and Mangku [3] consider the problem of estimating the cyclic component λ_c at a given point $s \in [0, \tau)$ of the intensity given in (1.1) of a cyclic Poisson process in the presence of linear trend. In fact, the estimator $\hat{\theta}_{n,b}$ given in (3.2) is used in [3] for correcting the bias of the estimator of λ_c. Estimation of the intensity function λ_c at a given point $s \in [0, \tau)$ of a purely cyclic Poisson process, that is Poisson proces having intensity given in (1.1) with $a = 0$, has been investigated, among others, in [4], [5], [6], [8], and [9].

There are many practical situations where we have to use only a single realization for estimating intensity of a cyclic Poisson process. A review of such applications can be seen in [4], and a number of them can also be found in [1], [7], [9], [11] and [12].

In section 2 we present the estimators and some preliminary results. These results are evaluated in section 3, by a Monte Carlo simulation. This evaluation leads to a bias corrected estimator. Proofs of all theorems are given in section 4.

2. The estimators and preliminary results

In this paper, we focus to the case when the period τ is known, but the slope a and the function λ_c on $[0, \tau)$ are both unknown. Note also that, in many practical applications, we know the period, for instance: one day, one week, one month, one year, etc. In this situation we may define estimators of respectively a and θ as follows

$$\hat{a}_n := \frac{2X(W_n)}{|W_n|^2}, \tag{2.1}$$

and

$$\hat{\theta}_n := \frac{1}{\ln(\frac{|W_n|}{\tau})} \sum_{k=1}^{\infty} \frac{1}{k} \frac{X([k\tau,(k+1)\tau] \cap W_n)}{\tau} - \hat{a}_n \left(\frac{\tau}{2} + \frac{|W_n|}{\ln(\frac{|W_n|}{\tau})} \right) \tag{2.2}$$

To obtain the estimator $\hat{a}_n$ of a it suffices to note that

$$\mathbf{E}X(W_n) = \frac{a}{2}|W_n|^2 + \mathcal{O}(|W_n|),$$

as $n \to \infty$, which directly yields the estimator given in (2.1). Note also that if X were a Poisson process with intensity $\lambda(s) = as$, then $\hat{a}_n$ would be the maximum likelihood estimator of a (cf. [10]).

Next, we describe the idea behind the construction of the estimator $\hat{\theta}_n$ of θ. For any $k \in \mathbf{N}$, we can write

$$\theta = \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \lambda_c(s) ds. \tag{2.3}$$

Let $L_n := \sum_{k=1}^{\infty} k^{-1} \mathbf{I}(k\tau \in W_n)$. Then, by (2.3), we can write

$$\begin{aligned}
\theta &= \frac{1}{L_n} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \lambda_c(s) \mathbf{I}(s \in W_n) ds \\
&= \frac{1}{L_n} \sum_{k=1}^{\infty} \frac{1}{k\tau} \int_{k\tau}^{(k+1)\tau} (\lambda(s) - as) \mathbf{I}(s \in W_n) ds \\
&= \frac{1}{L_n} \sum_{k=1}^{\infty} \frac{1}{k\tau} \int_{k\tau}^{(k+1)\tau} \lambda(s) \mathbf{I}(s \in W_n) ds \\
&\quad - \frac{a}{L_n} \sum_{k=1}^{\infty} \frac{1}{k\tau} \int_0^{\tau} (s + k\tau) \mathbf{I}(s + k\tau \in W_n) ds \\
&= \frac{1}{L_n} \sum_{k=1}^{\infty} \frac{1}{k} \frac{\mathbf{E}X([k\tau,(k+1)\tau] \cap W_n)}{\tau} \\
&\quad - \frac{a}{L_n \tau} \int_0^{\tau} s \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{I}(s + k\tau \in W_n) ds \\
&\quad - \frac{a}{L_n} \int_0^{\tau} \sum_{k=1}^{\infty} \mathbf{I}(s + k\tau \in W_n) ds. \qquad (2.4)
\end{aligned}$$

By noting that $\sum_{k=1}^{\infty} k^{-1} \mathbf{I}(s + k\tau \in W_n) = L_n + \mathcal{O}(1) \approx L_n$ and $\int_0^{\tau} s \; ds = \tau^2/2$, we see that the second term on the r.h.s. of (2.4) is

$\approx a\tau/2$. We also will use the fact that

$$\begin{aligned}\frac{a}{L_n}\int_0^\tau \sum_{k=1}^\infty \mathbf{I}(s+k\tau\in W_n)ds &= \frac{a\tau}{L_n}\left(\frac{|W_n|}{\tau}+\zeta_n\right) \\ &= \frac{a|W_n|}{L_n}+\frac{a\tau\zeta_n}{L_n}\approx \frac{a|W_n|}{L_n}\end{aligned}$$

(cf. definition of ζ_n in line below (2.10)), where $|\zeta_n|\leq 1$ for all $n\geq 1$. By these approximations and by approximating the expectation in the first term on the r.h.s. of (2.4) with its stochastic counterpart, we obtain

$$\theta \approx \frac{1}{L_n}\sum_{k=1}^\infty \frac{1}{k}\frac{\mathbf{E}X([k\tau,(k+1)\tau]\cap W_n)}{\tau} - a\left(\frac{\tau}{2}+\frac{|W_n|}{L_n}\right). \tag{2.5}$$

From the $\approx$ in (2.5) and noting that $L_n \sim \ln(|W_n|/\tau)$ as $n\to\infty$, we see that

$$\bar{\theta}_n = \frac{1}{\ln(\frac{|W_n|}{\tau})}\sum_{k=1}^\infty \frac{1}{k}\frac{\mathbf{E}X([k\tau,(k+1)\tau]\cap W_n)}{\tau} - a\left(\frac{\tau}{2}+\frac{|W_n|}{\ln(\frac{|W_n|}{\tau})}\right) \tag{2.6}$$

can be viewed as an estimator of θ, provided both the period τ and the slope a of the linear trend are assumed to be known. If a is unknown, we replace a by $\hat{a}_n$ (cf. (2.1)) and one obtains the estimator of θ given in (2.2).

In Helmers and Mangku [3] has been proved the following lemma.

Lemma 2.1. *Suppose that the intensity function λ satisfies (1.1) and is locally integrable. Then we have*

$$\mathbf{E}\left(\hat{a}_n\right) = a + \frac{2\theta}{|W_n|} + \mathcal{O}\left(\frac{1}{|W_n|^2}\right) \tag{2.7}$$

and

$$Var\left(\hat{a}_n\right) = \frac{2a}{|W_n|^2} + \mathcal{O}\left(\frac{1}{|W_n|^3}\right) \tag{2.8}$$

as $n\to\infty$. Hence, by (1.3), $\hat{a}_n$ is a consistent estimator of a; its mean-squared error (MSE) is given by $MSE(\hat{a}_n) = (4\theta^2+2a)|W_n|^{-2} + \mathcal{O}(|W_n|^{-3})$ as $n\to\infty$.

Consistency of $\hat{\theta}_n$ is established in Theorem 2.2. In Theorem 2.3 we compute the asymptotic approximations to respectively the bias and variance of the estimator $\hat{\theta}_n$.

Theorem 2.2. *Suppose that the intensity function λ satisfies (1.1) and is locally integrable. Then we have*

$$\hat{\theta}_n \xrightarrow{p} \theta, \tag{2.9}$$

as $n \to \infty$. In other words, $\hat{\theta}_n$ is a consistent estimator of θ. In addition, the MSE of $\hat{\theta}_n$ converges to 0, as $n \to \infty$.

Theorem 2.3. *Suppose that the intensity function λ satisfies (1.1) and is locally integrable. Then we have*

$$\mathbf{E}\hat{\theta}_n = \theta - \frac{(2-\gamma)\theta - (\gamma/2 + \zeta_n)a\tau}{\ln(|W_n|/\tau)} + o\left(\frac{1}{\ln|W_n|}\right) \tag{2.10}$$

as $n \to \infty$, where where $\gamma = 0.577..$ is Euler's constant and $\zeta_n = (\tau)^{-1}\int_0^\tau \sum_{k=1}^\infty \mathbf{I}(x + k\tau \in W_n)dx - (|W_n|/\tau)$ and $|\zeta_n| \le 1$ for all $n \ge 1$. In addition, we also have

$$\begin{aligned} Var\left(\hat{\theta}_n\right) &= \frac{a}{\ln(|W_n|/\tau)} + \frac{(\theta/\tau + a/2)(\pi^2/6) - a(2-\gamma)}{(\ln(|W_n|/\tau))^2} \\ &\quad + o\left(\frac{1}{(\ln|W_n|)^2}\right) \end{aligned} \tag{2.11}$$

as $n \to \infty$.

3. Simulations and bias reduction

For the simulations, we consider the intensity function

$$\lambda(s) = \lambda_c(s) + as = A\exp\left\{\rho\cos\left(\frac{2\pi s}{\tau} + \phi\right)\right\} + as,$$

that is (1.1), where λ_c is the intensity function discussed in Vere-Jones [14]. We chose $\rho = 1$, $\tau = 5$, $\phi = 0$ and $a = 0.05$. With this choice of the parameters, we have

$$\lambda(s) = A\exp\left\{\cos\left(\frac{2\pi s}{\tau}\right)\right\} + 0.05s. \tag{3.1}$$

In our simulations we consider three values of θ, which is determined by the choice of A, namely (i) a small value of θ, i.e. $\theta = 1.2661$ ($A = 1$), (ii) a moderate value of θ, i.e. $\theta = 2.5322$ ($A = 2$) and (iii) a large value of θ, i.e. $\theta = 5.0644$ ($A = 4$) (cf. Remark 3.4). We use $W_n = [0, 1000]$.

Example 3.1. In this example we study the performance of the estimator $\hat{\theta}_n$ in (2.2), in the case that the intensity function $\lambda(s)$ is given by (3.1).

(i) For the case small value of θ, ($\theta = 1.2661$), by (2.10) and (2.11), we obtain the asymptotic approximations to respectively the bias and the variance of $\hat{\theta}_n$ as follows: $Bias(\hat{\theta}_n) = -0.3736$ and $Var(\hat{\theta}_n) = 0.0232$. From the simulation, using $M = 10^4$ independent realizations of the process X observed in the $W_n = [0, 1000]$, we obtain respectively $\hat{Bias}(\hat{\theta}_n) = -0.3793$ and $\hat{V}ar(\hat{\lambda}_n) = 0.0221$, where $\hat{Bias}(\hat{\theta}_n)$ is the sample mean minus the true value θ and $\hat{V}ar(\hat{\theta}_n)$ is the sample variance. Summarizing, we have $Bias(\hat{\theta}_n)-$

$\hat{B}ias(\hat{\theta}_n) = -0.3736-(-0.3793) = 0.0057$ and $Var(\hat{\theta}_n)-\hat{V}ar(\hat{\lambda}_n) = 0.0232 - 0.0221 = 0.0011$.

(ii) For $\theta = 2.5322$, by (2.10) and (2.11), and from the simulation ($M = 10^4$) we obtain respectively $Bias(\hat{\theta}_n)-\hat{B}ias(\hat{\theta}_n) = -0.7137-(-0.7303) = 0.0166$ and
$Var(\hat{\theta}_n) - \hat{V}ar(\hat{\lambda}_n) = 0.0381 - 0.0364 = 0.0017$.

(iii) For $\theta = 5.0644$, by (2.10) and (2.11), and from the simulation ($M = 10^4$) we obtain respectively $Bias(\hat{\theta}_n)-\hat{B}ias(\hat{\theta}_n) = -1.3938-(-1.4210) = 0.0272$ and
$Var(\hat{\theta}_n) - \hat{V}ar(\hat{\lambda}_n) = 0.0677 - 0.0634 = 0.0043$.

From Example 3.1, we see that the asymptotic approximations to the bias and variance given in (2.10) and (2.11) predict quite well the variance and bias of the estimator $\hat{\theta}_n$ in finite samples. However, we see that the bias of $\hat{\theta}_n$ is quite big. We can reduce this bias by adding an estimator of the second term on the r.h.s. of (2.10) into $\hat{\theta}_n$. By employing this idea, we obtain a bias corrected estimator of θ as follows

$$\hat{\theta}_{n,b} := \hat{\theta}_n + \frac{(2-\gamma)\hat{\theta}_n - (\gamma/2 + \zeta_n)\tau\hat{a}_n}{\ln(|W_n|/\tau)}. \tag{3.2}$$

Theorem 3.2. *Suppose that the intensity function λ satisfies (1.1) and is locally integrable. Then we have*

$$\mathbf{E}\hat{\theta}_{n,b} = \theta + o\left(\frac{1}{\ln|W_n|}\right), \tag{3.3}$$

as $n \to \infty$, and

$$\begin{aligned} Var\left(\hat{\theta}_{n,b}\right) &= \frac{a}{\ln(|W_n|/\tau)} + \frac{(\theta/\tau + a/2)(\pi^2/6) + a(2-\gamma)}{(\ln(|W_n|/\tau))^2} \\ &\quad + o\left(\frac{1}{(\ln|W_n|)^2}\right), \end{aligned} \tag{3.4}$$

as $n \to \infty$.

Example 3.3. In this example we study the performance of the estimator $\hat{\theta}_{n,b}$ in (3.2), in the case that the intensity function $\lambda(s)$ is given by (3.1).

(i) For $\theta = 1.2661$, from the simulation ($M = 10^4$) and by (3.4), we obtain respectively $\hat{B}ias(\hat{\theta}_{n,b}) = -0.1090$ and $Var(\hat{\theta}_{n,b}) - \hat{V}ar(\hat{\theta}_{n,b}) = 0.0283 - 0.0354| = -0.0071$.

(ii) For $\theta = 2.5322$, from the simulation ($M = 10^4$) and by (3.4), we obtain respectively $\hat{B}ias(\hat{\theta}_{n,b}) = -0.2056$ and $Var(\hat{\theta}_{n,b}) - \hat{V}ar(\hat{\theta}_{n,b}) = 0.0431 - 0.0578 = -0.0147$.

(iii) For $\theta = 5.0644$, from the simulation ($M = 10^4$) and by (3.4), we obtain respectively $\hat{B}ias(\hat{\theta}_{n,b}) = -0.3993$ and $Var(\hat{\theta}_{n,b}) - \hat{V}ar(\hat{\theta}_{n,b}) = 0.0728 - 0.1051 = -0.0323$.

It is clear that the bias of the estimator $\hat{\theta}_{n,b}$ is much smaller than the bias of the original estimator $\hat{\theta}_n$. So the bias reduction proposed in (3.2) works.

Remark 3.4. A cautionary remark on the range of validity of Theorems 2.3 and 3.2 in practical applications is important here. The remainder terms in (2.10), (2.11), (3.3) and (3.4) will depend on the values of the parameters involved, such as θ, a, and τ. In order to have the approximations in (2.10) and (3.3) to be valid, θ and $a\tau$ should not too big compared to $\ln(|W_n|/\tau)$. The second order approximations in (2.11) and (3.4) is valid provided θ and $a\tau$ are not too big compared to $(\ln(|W_n|/\tau))^2$. Note that, in case (iii) of Examples 3.1 and 3.3 we have $\theta = 5.0644$ which is almost the same as the value of $\ln(|W_n|/\tau) = 5.2983$. For this reason, we do not consider the case where the value of θ is larger than that in case (iii) of the current examples.

To conclude this section, we remark that the bias corrected estimator $\hat{\theta}_{n,b}$ is to be preferred in practical applications.

4. Proofs

We first prove Theorem 2.3.

Proof of Theorem 2.3

First we prove (2.10). Note that

$$\begin{aligned}\mathbf{E}\hat{\theta}_n &= \frac{1}{\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\frac{1}{k}\frac{\mathbf{E}X([k\tau,(k+1)\tau]\cap W_n)}{\tau} \\ &\quad -\left(\frac{\tau}{2}+\frac{|W_n|}{\ln(|W_n|/\tau)}\right)\mathbf{E}\hat{a}_n. \end{aligned} \tag{4.1}$$

The first term on the r.h.s. of (4.1) is equal to

$$\begin{aligned} &\frac{1}{\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\frac{1}{k\tau}\int_{k\tau}^{k\tau+\tau}\lambda(x)\mathbf{I}(x\in W_n)dx \\ =\ &\frac{1}{\ln(|W_n|/\tau)}\sum_{k=1}^{\infty}\frac{1}{k\tau}\int_0^{\tau}\left(\lambda_c(x+k\tau)+a(x+k\tau)\right)\mathbf{I}(x+k\tau\in W_n)dx \\ =\ &\frac{1}{\ln(|W_n|/\tau)}\frac{1}{\tau}\int_0^{\tau}\lambda_c(x)\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{I}(x+k\tau\in W_n)dx \\ &+\frac{a}{\ln(|W_n|/\tau)}\frac{1}{\tau}\int_0^{\tau}x\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{I}(x+k\tau\in W_n)dx \\ &+\frac{a}{\ln(|W_n|/\tau)}\frac{1}{\tau}\int_0^{\tau}\sum_{k=1}^{\infty}\mathbf{I}(x+k\tau\in W_n)dx, \end{aligned} \tag{4.2}$$

where we have used (1.1) and (1.2). Note that

$$\sum_{k=1}^{\infty} k^{-1}\mathbf{I}(x+k\tau\in W_n)=\ln(|W_n|/\tau)+\gamma+o(1) \tag{4.3}$$

as $n\to\infty$ uniformly in $x\in[0,\tau]$ (cf. [13], p.150). Using (4.3), a simple calculation shows that the first term on the r.h.s. of (4.2) is equal to

$$\theta+\frac{\theta\gamma}{\ln(|W_n|/\tau)}+o\left(\frac{1}{\ln|W_n|}\right) \tag{4.4}$$

as $n\to\infty$, and its second term is equal to

$$\frac{a\tau}{2}+\frac{a\tau\gamma/2}{\ln(|W_n|/\tau)}+o\left(\frac{1}{\ln|W_n|}\right) \tag{4.5}$$

as $n\to\infty$. Clearly

$$\frac{1}{\tau}\int_0^{\tau}\sum_{k=1}^{\infty}\mathbf{I}(x+k\tau\in W_n)dx=|W_n|/\tau+\zeta_n \tag{4.6}$$

(cf. definition of ζ_n in line below (2.10)). By (4.6), the third term on the r.h.s. of (4.2) reduces to

$$\frac{a|W_n|}{\ln(|W_n|/\tau)}+\frac{a\tau\zeta_n}{\ln(|W_n|/\tau)}. \tag{4.7}$$

Using (2.7), the second term on the r.h.s. of (4.1) reduces to

$$-\frac{a\tau}{2}-\frac{a|W_n|}{\ln(|W_n|/\tau)}-\frac{2\theta}{\ln(|W_n|/\tau)}+\mathcal{O}\left(\frac{1}{|W_n|}\right) \tag{4.8}$$

as $n\to\infty$. Combining (4.4), (4.5), (4.7) and (4.8), we obtain (2.10).

Next we prove (2.11). Let A_n and $-B_n$ denote respectively the first and second term on the r.h.s. of (2.2). In other words, we write $\hat{\theta}_n=A_n-B_n$. Then we can compute the variance of $\hat{\theta}_n$ as follows

$$Var\left(\hat{\theta}_n\right)=Var\left(A_n\right)+Var\left(B_n\right)-2Cov\left(A_n,B_n\right). \tag{4.9}$$

Note that, for any $j\neq k$, $j,k=1,2,\dots$, we have $X([j\tau,(j+1)\tau]\cap W_n)$ and $X([k\tau,(k+1)\tau]\cap W_n)$ are independent. Then $Var(A_n)$ can be computed as follows

$$
\begin{aligned}
& Var\,(A_n) \\
= \;& \frac{1}{(\ln(|W_n|/\tau))^2}\sum_{k=1}^{\infty}\frac{1}{k^2\tau^2}Var\,(X([k\tau,(k+1)\tau]\cap W_n)) \\
= \;& \frac{1}{\tau^2(\ln(|W_n|/\tau))^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\tau}(\lambda_c(x)+a(x+k\tau))\,\mathbf{I}(x+k\tau\in W_n)dx \\
= \;& \frac{1}{\tau^2(\ln(|W_n|/\tau))^2}\int_0^{\tau}\lambda_c(x)\sum_{k=1}^{\infty}\frac{1}{k^2}\mathbf{I}(x+k\tau\in W_n)dx \\
& +\frac{a}{\tau^2(\ln(|W_n|/\tau))^2}\int_0^{\tau}x\sum_{k=1}^{\infty}\frac{1}{k^2}\mathbf{I}(x+k\tau\in W_n)dx \\
& +\frac{a}{\tau(\ln(|W_n|/\tau))^2}\int_0^{\tau}\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{I}(x+k\tau\in W_n)dx. \qquad (4.10)
\end{aligned}
$$

Here we have used (1.1) and (1.2). Note that

$$
\sum_{k=1}^{\infty}\frac{1}{k^2}\mathbf{I}(x+k\tau\in W_n)=\frac{\pi^2}{6}+o(1) \qquad (4.11)
$$

as $n\to\infty$, uniformly in $x\in[0,\tau]$ (cf. [13], p.34). Using (4.11), a simple calculation shows that the first term on the r.h.s. of (4.10) is equal to

$$
\frac{(\theta/\tau)(\pi^2/6)}{(\ln(|W_n|/\tau))^2}+o\left(\frac{1}{(\ln|W_n|)^2}\right) \qquad (4.12)
$$

as $n\to\infty$, and its second term is equal to

$$
\frac{(a/2)(\pi^2/6)}{(\ln(|W_n|/\tau))^2}+o\left(\frac{1}{(\ln|W_n|)^2}\right) \qquad (4.13)
$$

as $n\to\infty$. By (4.3), the third term on the r.h.s. of (4.10) reduces to

$$
\frac{a}{\ln(|W_n|/\tau)}+\frac{a\gamma}{(\ln(|W_n|/\tau))^2}+o\left(\frac{1}{(\ln|W_n|)^2}\right) \qquad (4.14)
$$

as $n\to\infty$. Combining (4.12), (4.13) and (4.14), we obtain

$$
Var\,(A_n)=\frac{a}{\ln(|W_n|/\tau)}+\frac{(\theta/\tau+a/2)\pi^2/6+a\gamma}{(\ln(|W_n|/\tau))^2}+o\left(\frac{1}{(\ln|W_n|)^2}\right) \qquad (4.15)
$$

as $n\to\infty$.

Next we consider the second and third term on the r.h.s. of (4.9). By (2.8), we obtain

$$
\begin{aligned}
Var(B_n) &= \left(\tau/2 + \frac{|W_n|}{\ln(|W_n|/\tau)}\right)^2 \left(\frac{2a}{|W_n|^2} + \mathcal{O}\left(\frac{1}{|W_n|^3}\right)\right) \\
&= \frac{2a}{(\ln(|W_n|/\tau))^2} + \mathcal{O}\left(\frac{1}{|W_n|\ln(|W_n|/\tau)}\right) \qquad (4.16)
\end{aligned}
$$

as $n \to \infty$. Next we compute $Cov(A_n, B_n)$ as follows

$$
\begin{aligned}
&Cov(A_n, B_n) \\
= &\left(\frac{|W_n|}{\ln(\frac{|W_n|}{\tau})} + \tau/2\right)\left(\frac{2}{|W_n|^2\ln(\frac{|W_n|}{\tau})}\right) \\
&\sum_{k=1}^{\infty}\frac{1}{k\tau}Cov\left(X([k\tau,(k+1)\tau]\cap W_n), X(W_n)\right) \\
= &\left(\frac{2}{\tau|W_n|(\ln(\frac{|W_n|}{\tau}))^2} + \frac{1}{|W_n|^2\ln(\frac{|W_n|}{\tau})}\right) \\
&\sum_{k=1}^{\infty}\frac{1}{k}Var\left(X([k\tau,(k+1)\tau]\cap W_n)\right) \\
= &\left(\frac{2}{\tau|W_n|(\ln(\frac{|W_n|}{\tau}))^2} + \frac{1}{|W_n|^2\ln(\frac{|W_n|}{\tau})}\right) \\
&\int_0^{\tau}(\lambda_c(x) + ax)\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{I}(x + k\tau \in W_n)dx \\
&+ \left(\frac{2}{\tau|W_n|(\ln(\frac{|W_n|}{\tau}))^2} + \frac{1}{|W_n|^2\ln(\frac{|W_n|}{\tau})}\right)a\tau\int_0^{\tau}\sum_{k=1}^{\infty}\mathbf{I}(x + k\tau \in W_n)dx.
\end{aligned}
\qquad (4.17)
$$

By (4.3), we see that the first term on the r.h.s. of (4.17) is of order $\mathcal{O}(|W_n|^{-1}(\ln|W_n|)^{-1})$, as $n \to \infty$. By (4.6), the second term on the r.h.s. of (4.17) reduces to $2a(\ln|W_n|)^{-2} + \mathcal{O}(|W_n|^{-1}(\ln|W_n|)^{-1})$, as $n \to \infty$. Hence, the third term on the r.h.s. of (4.9) is equal to

$$
-2Cov(A_n, B_n) = -\frac{4a}{(\ln(|W_n|/\tau))^2} + \mathcal{O}\left(\frac{1}{|W_n|\ln(|W_n|/\tau)}\right) \qquad (4.18)
$$

as $n \to \infty$. Combining (4.15), (4.16) and (4.18), we obtain (2.11). This completes the proof of Theorem 2.3.

Proof of Theorem 2.2

By (2.10) and the assumption (1.3), we obtain

$$
\mathbf{E}\hat{\theta}_n = \theta + o(1) \qquad (4.19)
$$

as $n \to \infty$, while (2.11) and assumption (1.3) imply

$$Var\left(\hat{\theta}_n\right) = o(1) \tag{4.20}$$

as $n \to \infty$. Together (4.19) and (4.20), imply (2.9). This completes the proof of Theorem 2.2.

Proof of Theorem 3.2

First we prove (3.3). To do this, we first rewrite the estimator $\hat{\theta}_{n,b}$ in (3.2)

$$\hat{\theta}_{n,b} := \left(1 + \frac{(2-\gamma)}{\ln(|W_n|/\tau)}\right)\hat{\theta}_n - \frac{(\gamma/2+\zeta_n)\tau}{\ln(|W_n|/\tau)}\hat{a}_n. \tag{4.21}$$

By (2.10), we see that the expectation of the first term on the r.h.s. of (4.21) is equal to

$$\theta + \frac{(\gamma/2+\zeta_n)a\tau}{\ln(|W_n|/\tau)} + o\left(\frac{1}{\ln|W_n|}\right) \tag{4.22}$$

as $n \to \infty$. By (2.7), the expectation of the second term on the r.h.s. of (4.21) reduces to

$$-\frac{(\gamma/2+\zeta_n)\tau a}{\ln(|W_n|/\tau)} + o\left(\frac{1}{\ln|W_n|}\right) \tag{4.23}$$

as $n \to \infty$. Combining (4.22) and (4.23) we obtain (3.3).

Next we prove (3.4). Using (4.21), $Var(\hat{\theta}_{n,b})$ can be computed as follows

$$\begin{aligned} Var(\hat{\theta}_{n,b}) &= \left(1 + \frac{(2-\gamma)}{\ln(|W_n|/\tau)}\right)^2 Var(\hat{\theta}_n) + \frac{(\gamma/2+\zeta_n)^2\tau^2}{(\ln(|W_n|/\tau))^2} Var(\hat{a}_n) \\ &\quad -2\left(1 + \frac{(2-\gamma)}{\ln(|W_n|/\tau)}\right)\frac{(\gamma/2+\zeta_n)\tau}{\ln(|W_n|/\tau)} Cov(\hat{\theta}_n, \hat{a}_n). \end{aligned} \tag{4.24}$$

By (2.11), we see that the first term on the r.h.s. of (4.24) is equal to

$$\begin{aligned} &\frac{a}{\ln(|W_n|/\tau)} + \frac{(\theta/\tau + a/2)(\pi^2/6) - a(2-\gamma)}{(\ln(|W_n|/\tau))^2} + \frac{2a(2-\gamma)}{(\ln(|W_n|/\tau))^2} \\ &+o\left(\frac{1}{(\ln|W_n|)^2}\right) \\ &= \frac{a}{\ln(|W_n|/\tau)} + \frac{(\theta/\tau + a/2)(\pi^2/6) + a(2-\gamma)}{(\ln(|W_n|/\tau))^2} + o\left(\frac{1}{(\ln|W_n|)^2}\right) \end{aligned} \tag{4.25}$$

as $n \to \infty$. Note that, the $-a(2-\gamma)$ on the r.h.s. of (2.11) is replaced by $+a(2-\gamma)$ on the r.h.s. of (4.25). By (2.8), it easily seen that the second term on the r.h.s. of (4.24) is of order $\mathcal{O}(|W_n|^{-2}(\ln|W_n))^{-2})$, which is $o((\ln|W_n))^{-2})$ as $n \to \infty$. Finally, Cauchy-Schwarz inequality shows that the third term on the r.h.s. of (4.24) is of negligible order $o((\ln|W_n|)^{-2})$ as $n \to \infty$. Hence, we have (3.4). This completes the proof of Theorem 3.2.

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