AN INVESTMENT STRATEGY IN PORTFOLIO SELECTION PROBLEM WITH BULLET TRANSACTION COST

EFFENDI SYAHRIL

Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Bogor Agricultural University
Jl. Raya Pajajaran, Kampus IPB Baranangsiang, Bogor, Indonesia

Abstract. This paper discusses an investment strategy for a consumption and investment decision problem for an individual who has available a riskless asset paying fixed interest rate and a risky asset driven by Brownian motion price fluctuations. The individual observes current wealth when making transactions, that transactions incur costs, and that decisions to transact can be made at any time based on all current information. The transactions costs is fixed for every transaction, regardless of amount transacted. In addition, the investor is charged a fixed fraction of total wealth as management fee. The investor's objective is to maximize the expected utility of consumption over a given horizon. The problem faced by the investor is formulated in a stochastic discrete-continuous-time control problem. An investment strategy is given for fixed transaction intervals.
Key words: Investment strategy, fixed transaction cost, continuous-discrete-time, stochastic optimal control problem

1. Introduction

The publication of Merton's seminal work, see Merton(1971), has started the application of stochastic optimal control and stochastic calculus techniques to the area of finance. Merton (1971, 1990) studied the behaviour of a single agent acting as a market price-taker who seeks to maximize expected utility of consumption. The utility function of the agent was assumed to be a power function, and the market was assumed to comprise a risk-free asset with constant rate of return and one or more stocks, each with constant mean rate of return and volatility. The only information available to the agent were current prices of the assets. There were no transaction costs. It was also assumed that the assets were divisible. In this idealized setting, Merton was able to derive a closed-form solution to the stochastic optimal control problem faced by the agent.

Several authors have made contributions to the stochastic optimal control and stochastic calculus analyses of the Merton's model. To mention a few among them are Constantinides (1979, 1986), Cox and Huang (1989), Davis and Norman (1990), Duffie and Sun (1990), Magill and Constantinides (1976).

The application of transaction costs to Merton's model was first accomplished by Magill and Constantinides (1976). Several authors then have published a number of works on Merton's model with transaction costs. To mention a few, they are Constantinides (1979, 1986), Davis and Norman (1990), Duffie and Sun (1990). Duffie and Sun (1990) treated the proportional transaction costs with different formulation to others, which they call *discrete-continuous-time formulation.* Their formulation assumes that an investor observes current wealth when making transaction, and decisions to transact can be made at any time, but without no costs. They treated general linear transaction costs of the form $a\,W_{\tau_n} + b$, with W_{τ_n} denotes the amount of wealth transacted, and a and b are non-negatives. Based on work of Duffie and Sun (1990), Syahril (2003) re-writes *discrete-continuous-time* formulation of Merton's model with fixed transaction cost. This paper investigates an optimal investment strategies for *discrete-continuous-time* formulation of Merton's model with fixed transaction cost.

2. Formulation of the Model

2.1. **Securities Market and Transactions.** It is assumed that a complete probability space $(\Omega, \mathcal{F}, P)$ is given. In addition, it is assumed that *a filtration* $\{\mathcal{F}_t : t \geq 0\}$ is also given. By a filtration is meant a family of σ- algebras $\{\mathcal{F}_t : t \geq 0\}$ which is increasing : $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. It is assumed that the one-dimensional standard Brownian motion $B = \{B_t : t \geq 0\}$ is given on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, P)$.
There are two securities available in the economy to an investor. One is a riskless security with fixed interest rate r, and the other is a risky security whose price is a geometric Brownian motion with expected rate of return α and rate of return variation σ^2. At time $t \geq 0$, the price processes $\{P_0(t)\}$ of the riskless security satisfy a deterministic differential equation

$$dP_0(t) = rP_0(t)dt, \tag{2.1}$$

while the price processes $\{P_1(t)\}$ of the risky security satisfy a stochastic differential equation

$$dP_1(t) = \alpha P_1(t)dt + \sigma P_1(t)dB_t. \tag{2.2}$$

Money is available for the investor in the economy as a medium of exchange and numeraire. Only money is exchangable for consumption. Let M_t denotes money holdings at time t. The investor is assumed to receive no further income from noncapital sources, and starts with the initial stock of money $M_0 = 0$. Trading opportunities are available continuously in time, but with costs. Transactions costs are incurred when information is processed and a portfolio transaction is made. There are two forms of transaction costs: portfolio management fees and withdrawal costs. The investor pays a fraction $\varepsilon > 0$ of the total wealth in the securities at the beginning of each interval as a portfolio management fee. The portfolio management fee is meant to include the cost of adjusting the portfolio and the cost of processing information. In this paper, transactions costs is the costs which incurs during withdrawing wealth from the portfolio. The transaction costs is a fixed for every transaction, regardless of amount of wealth transacted. Then the total transaction costs function is of the form $b + \varepsilon(X_{\tau_n} - W_{\tau_n})$, where X_{τ_n} is the total wealth at time τ_n before transaction. Filtration $(\mathcal{F}_t)$ defined by $\mathcal{F}_t = \sigma\{B_s : s \leq t\}$, will be interpreted as information available up to time t. Given the structure of transaction costs, consumption and investment decisions are made at intervals. During each interval there is no transaction. All dividends of risky security are re-invested continually in the risky security, and all interest income is re-invested continually in the riskless security.

The investor chooses instants of time at which to process information and make consumption and investment decisions. In other words, information is available continuously through the filtration $\{\mathcal{F}_t : t \geq 0\}$. The investor receives information via controllable filtration

$$\mathbf{H} = \{\mathcal{H}_t : t \geq 0\} \text{ with } \mathcal{H}_t = \mathcal{F}_t,\ t \in [\tau_n, \tau_{n+1}),$$

where τ_n is a $\mathcal{H}_{\tau_{n-1}}$-measurable stopping time at which the n-th transaction occurs. The filtration $\mathbf{H}$ is controllable in the sense that the investor is allowed to choose any sequence $\tau = \{\tau_n : n = 1, 2, 3, ...\}$ of such transaction times with $\tau_1 \equiv 0$. Let $T = \{T_n = \tau_{n+1} - \tau_n : n = 1, 2, 3, ...\}$ denotes the corresponding sequence of transaction intervals. Finding an optimal stopping policy τ is clearly equivalent to finding an optimal transaction interval policy T.

2.2. **Formulation of The Model.** Let the consumption space $\mathcal{C}$ for the investor consists of positive $\mathbf{H}$-adapted consumption processes $\mathbf{C} = \{C_t : t \geq 0\}$ satisfying $\int_0^t C_s ds < \infty$ almost surely for all $t \geq 0$, and

$$E[\int_0^\infty e^{-\delta\, t} u(C_t) dt] < \infty, \tag{2.3}$$

where E denotes the *expected value function*, with respect to P, δ is a strictly positive scalar discount factor and the utility function u, is one of the HARA (hyperbolic absolute risk-aversion) type function, as defined in Merton (1971). We take u as given by

$$u(C) = \frac{1}{\gamma} C^{\gamma}, \quad 0 < \gamma < 1. \tag{2.4}$$

Let $\tau = \{\tau_n : n = 1, 2, 3, ...\}$ be sequence of transaction times with $\tau_1 \equiv 0$. Let $T = \{T_n = \tau_{n+1} - \tau_n, n = 1, 2, 3, ...\}$ be the sequence of corresponding transaction intervals. Let $W = \{W_{\tau_n} : n = 1, 2, 3, ...\}$ be the sequence of money withdrawal processes, and $V = \{V_{\tau_n} : n = 1, 2, 3, ...\}$ be the sequence of investment for the risky security.

Let $\mathcal{T}$ denote the space of sequences of strictly positive transaction intervals, $\mathcal{W}$ the space of positive $\mathbf{H}$-adapted money withdrawal processes, and $\mathcal{V}$ the space of $\mathbf{H}$-adapted investment processes for the risky security. Let $\mathcal{U} = \mathcal{T} \times \mathcal{W} \times \mathcal{V} \times \mathcal{C}$.

Definition 2.1. A budget policy is a quadruplet $(T, W, V, C) \in \mathcal{U}$.

We characterize budget feasible policies as follows. Let $\mathcal{U}$ denotes a class of budget policies. Given a policy $(T, W, V, C) \in \mathcal{U}$, then the money holding at any time t is defined by

$$M_t = \sum_{\{n:\tau_n \le t\}} [\, W_{\tau_n} - b\,] - \int_0^t C_s \, ds, \tag{2.5}$$

Let X_{τ_n} denotes the total wealth invested in the securities at time τ_n, before the nth transaction. Let W_{τ_n} denotes the amount of money withdrawn at time τ_n from the total wealth X_{τ_n}, and V_{τ_n} denotes the market value of the investment in the risky security chosen at time τ_n. After an amount W_{τ_n} is withdrawn from the total wealth X_{τ_n}, and a fraction ε of the remainder, is paid as management fees, then the wealth left for re-investment is $Z_{\tau_n} = (1 - \varepsilon)\, [\, X_{\tau_n} - W_{\tau_n}\,]$. Of this amount, V_{τ_n} is invested in the risky security with a per-dollar payback of Γ_{n+1} at the next transaction date, including continually re-invested dividends. And the remainder, $Z_{\tau_n} - V_{\tau_n}$, is invested in the riskless security at the continuously compounding interest rate $r > 0$.

The investor's total wealth invested at the time of the $(n+1)$th transaction is therefore

$$X_{\tau_{n+1}} = (\, 1 - \varepsilon\,)\, [\, X_{\tau_n} - W_{\tau_n}\,]\, e^{r\,T_n} + V_{\tau_n}\, [\, \Gamma_{n+1} - e^{r\,T_n}\,]. \tag{2.6}$$

for $n = 1, 2, 3, ...,$

According to the equation (2.2) and the Itô's formula,[1] the return of the risky investment Γ satisfies

$$\Gamma_{n+1} = \exp\,[(\alpha - \frac{1}{2}\,\sigma^2)\, T_n + \sigma\,(B_{\tau_{n+1}} - B_{\tau_n})]. \tag{2.7}$$

Since $M_0 = 0$, then X_0 is considered as the initial wealth endowment for the investor.

[1]Details may be found in Karatzas and Shreve (1988), or Protter (1990)

Definition 2.2. The budget policy $(T, W, V, C) \in \mathcal{U}$ is budget feasible policy if the associated money process M of (2.5) and invested wealth process X of (2.6) are non-negative.

2.3. Optimal Control Statement of the Problem.

Definition 2.3. Let $\mathcal{U}$ be the set of all budget feasible policies as defined previously. The optimal control problem for the investor is to maximize

$$U(X_0) \equiv \max_{(T,W,V,C)\in\mathcal{U}} E\,[\int_0^{\infty} e^{-\delta t}\, u(C_t)\, dt], \tag{2.8}$$

subject to, for $n = 1, 2, 3, ...,$

$$X_{\tau_{n+1}} = (\,1 - \varepsilon\,)\,[\,X_{\tau_n} - W_{\tau_n}\,]\, e^{r\,T_n} + V_{\tau_n}\,[\,\Gamma_{n+1} - e^{r\,T_n}\,], \tag{2.9}$$

with $M_t \geq 0$, and $X_{\tau_{n+1}} \geq 0$.

We assume that only money is available to the investor as a medium of exchange and numeraire in the economy. Only money is exchangeable for consumption. It is also assumed that money cannot be borrowed, it can only be acquired by selling the securities, and it is put in the purse M. Because there exists a riskless security with a positive interest rate in the economy, there is no investment demand for money. Duffie and Sun (1990) argued that it will not be optimal for the investor to withdraw more money than the amount needed for financing consumption before the next transaction.

The following result is similar to those in Duffie and Sun (1990), the proof can be found in Duffie and Sun (1990) or in Syahril (2003).

Theorem 2.4. *Let the value function U be defined as in (2.8), and the transaction costs function $\Psi(W_{\tau_n}) = b$, $b \geq 0$. Then the optimal policy (T, W, V, C) must satisfy for all $n = 1, 2, 3, ...$*

$$\int_{\tau_n}^{\tau_{n+1}} C_t\, dt = W_{\tau_n} - b. \tag{2.10}$$

Corollary 1. By the definition of money holding M_t of equation (2.5), then

$$M_{\tau_n} = W_{\tau_n} - b, \quad n = 1, 2, 3, ...$$

Therefore, the optimal control problem (2.8)-(2.9) is equivalent to the optimal control problem :

$$U(X_0) = \max_{(T,W,V,C)\in\mathcal{U}} E\,[\,\int_0^{\infty} e^{-\delta\, t}\, u(C_t)\, dt\,] \tag{2.11}$$

subject to

$$\int_{\tau_n}^{\tau_{n+1}} C_t\, dt = W_{\tau_n} - b, \tag{2.12}$$

$$X_{\tau_{n+1}} = (1-\varepsilon)\,[\,X_{\tau_n} - W_{\tau_n}\,]\,e^{r\,T_n} + V_{\tau_n}\,[\,\Gamma_{n+1} - e^{r\,T_n}\,] \geq 0, \quad (2.13)$$

for $n = 1, 2, 3, ...$

We summarize the problem faced by the investor in the following definition. For the complete formulation, one can consult [2]

Definition 2.5. Let $\mathcal{U}$ be the set of all budget feasible policies as defined previously. The optimal control problem for the investor is to maximize

$$U(X_{\tau_n}) = \max_{\{T_n, W_{\tau_n}, V_{\tau_n}\}} \{Q_n^{\nu}\,\frac{1}{\gamma}\,(W_{\tau_n} - b)^{\gamma} + e^{-\delta T_n}\,E\,[U(X_{\tau_{n+1}}) \mid \mathcal{H}_{\tau_n}]\}, \quad (2.14)$$

subject to

$$X_{\tau_{n+1}} = (\,1-\varepsilon\,)\,[\,X_{\tau_n} - W_{\tau_n}\,]\,e^{r\,T_n} + V_{\tau_n}\,[\,\Gamma_{n+1} - e^{r\,T_n}\,], \quad (2.15)$$

for $n = 1, 2, 3, ...,$ with $M_t \geq 0,$ and $X_{\tau_{n+1}} \geq 0.$

3. **Investment Strategies**

The following result is similar to those in Hakansson(1970), and its proof is omitted.

Lemma 3.1. *Let* $u,\ \ \Gamma_{n+1},\ \ r,$ *and* T_n *be defined as previously. Then the function*

$$f(\pi) \equiv E\,[\,u(e^{r\,T_n} + \pi\,(\Gamma_{n+1} - e^{r\,T_n}))\,] \quad (3.1)$$

subject to the constraints $\pi \geq 0,$ *and*

$$P\,\{\,e^{r\,T_n} + \pi\,(\Gamma_{n+1} - e^{r\,T_n}) \geq 0\,\} = 1, \quad (3.2)$$

has a finite maximum.

Lemma 3.2. *Let* $\pi \geq 0,$ *then* $P\,\{e^{r\,T_n} + \pi\,(\Gamma_{n+1} - e^{r\,T_n}) \geq 0\} = 1$ *holds if and only if* $\pi \leq 1.$

Remark 3.3. Below maximizing with $0 \leq \pi \leq 1$ is equivalent to maximizing with $\pi \geq 0$ and (3.2).

Proof : Let $P\,\{e^{r\,T_n} + \pi\,(\Gamma_{n+1} - e^{r\,T_n}) \geq 0\} = 1.$ Then it will be shown that $\pi \leq 1.$
If $\pi > 1,$ it will be shown that

$$P\,\{e^{rT_n} + \pi(\Gamma_{n+1} - e^{rT_n}) \geq 0\} < 1.$$

Note that

$$e^{rT_n} + \pi(\Gamma_{n+1} - e^{rT_n}) \geq 0 \iff \Gamma_{n+1} \geq e^{rT_n}\,(1 - \frac{1}{\pi}).$$

---
[2] Syahril (2003)

Let $\lambda = \alpha - \frac{1}{2}\sigma^2$. Then

$$\begin{aligned} P\{\Gamma_{n+1} \geq e^{rT_n}\,(1-\frac{1}{\pi})\} &= P\{\lambda T_n + \sigma B_{T_n} > rT_n + \log(1-\frac{1}{\pi})\} \\ &= P\{B_{T_n} > \frac{1}{\sigma}(r-\lambda)T_n + \frac{1}{\sigma}\log(1-\frac{1}{\pi})\}. \end{aligned}$$

Let $l = \frac{1}{\sigma}\log(1-1/\pi)$, and $m = \frac{1}{\sigma}(r-\alpha+\frac{1}{2}\sigma^2)$. Then,

$$\begin{aligned} P\{B_{T_n} \geq l + mT_n\} &= \int_{l+mT_n}^{\infty} e^{-\frac{\zeta^2}{2}T_n}\,\frac{1}{\sqrt{2\pi T_n}}\,d\zeta \\ &= \int_{\frac{l}{\sqrt{T_n}}+m\sqrt{T_n}}^{\infty} e^{-\frac{y^2}{2}}\,\frac{1}{\sqrt{2\pi}}\,dy \\ &< 1. \end{aligned}$$

On the other hand, if $\pi \leq 1$, then

$$e^{rT_n} + \pi(\Gamma_{n+1} - e^{rT_n}) = (1-\pi)e^{rT_n} + \pi\Gamma_{n+1} \geq 0. \quad \blacksquare$$

Lemma 3.4. *Let $\delta > \max(\gamma\alpha, \gamma r)$, and let Ω_n be defined by*

$$\begin{aligned} \Omega_n &\equiv E\left([e^{r\,T_n} + \pi_n\,(\Gamma_{n+1} - e^{r\,T_n})]^{\gamma}\right) \\ &\equiv \sup_{\{0\leq\pi\leq 1\}} E\left([e^{r\,T_n} + \pi\,(\Gamma_{n+1} - e^{r\,T_n})]^{\gamma}\right). \end{aligned}$$

Then $R_n = (1-\varepsilon)^{\gamma}\,e^{-\delta T_n}\,\Omega_n$ is such that $R_n \leq (1-\varepsilon)^{\gamma}$.

Proof : Since $0 < \varepsilon < 1$ and $\pi_n \in [0,1]$, then

$$\begin{aligned} R_n &= (1-\varepsilon)^{\gamma}\,e^{-\delta T_n}\,\Omega_n \\ &= (1-\varepsilon)^{\gamma} e^{-\delta T_n}\,E\left([e^{rT_n} + \pi_n\,(\Gamma_{n+1} - e^{rT_n})]^{\gamma}\right) \\ &\leq (1-\varepsilon)^{\gamma} e^{-\delta T_n}\,[e^{rT_n} + \pi_n\,(E(\Gamma_{n+1}) - e^{rT_n})]^{\gamma} \\ &\leq (1-\varepsilon)^{\gamma} e^{-\delta T_n}\,[e^{rT_n} + \pi_n\,(e^{\alpha T_n} - e^{rT_n})]^{\gamma} \\ &\leq (1-\varepsilon)^{\gamma} e^{-\delta T_n}\,[e^{rT_n} + 1_{\{\alpha\geq r\}}\,(e^{\alpha T_n} - e^{rT_n})]^{\gamma} \\ &\leq (1-\varepsilon)^{\gamma} e^{-\delta T_n}\,e^{\gamma r T_n + \gamma\,1_{\{\alpha\geq r\}}\,(\alpha-r)T_n} \\ &\leq (1-\varepsilon)^{\gamma} e^{-\delta T_n}\,e^{\max(\gamma\alpha,\gamma r)T_n}. \end{aligned}$$

Since $\delta > \max(\gamma\alpha, \gamma r)$ by assumption, then $e^{-\delta T_n}\,e^{\max(\gamma\alpha,\gamma r)T_n} < 1$.
Therefore, $R_n = (1-\varepsilon)^{\gamma}\,e^{-\delta T_n}\,\Omega_n \leq (1-\varepsilon)^{\gamma}$. $\blacksquare$

Theorem 3.5. *Let T_n be fixed for $n = 1, 2, 3, ...$ Then the optimal value function and unique solution to problem (2.14)- (2.15), is given by*

$$U(X_{\tau_n}) = Q_n^{\nu}\,A_n^{-\nu}\,\frac{1}{\gamma}\,(X_{\tau_n} - Y_n)^{\gamma} \tag{3.3}$$

with the optimal withdrawal and investment strategies are given by

$$W_{\tau_n} = A_n\,(X_{\tau_n} - Y_n) + b \tag{3.4}$$

$$V_{\tau_n} = (1-\varepsilon)\,(1-A_n)\,(X_{\tau_n} - Y_n)\,\pi_n, \tag{3.5}$$

respectively, and where A_n, *and* Y_n *are given by*

$$A_n = \frac{A_{n+1}\, Q_n}{A_{n+1}\, Q_n + Q_{n+1}\, R_n^{1/\nu}} \tag{3.6}$$

$$Y_n = b + (1-\varepsilon)^{-1}\, e^{-r\,T_n}\, Y_{n+1} \tag{3.7}$$

respectively, with $R_n = (1-\varepsilon)^{\gamma}\, e^{-\delta\,T_n}\, \Omega_n$, *and where* Ω_n *and* π_n *are defined by the optimization problem*

$$\begin{aligned} \Omega_n &\equiv E\,(\,[\,e^{r\,T_n} + \pi_n\,(\Gamma_{n+1} - e^{r\,T_n})\,]^{\gamma}\,) \\ &\equiv \sup_{\{0\le\pi\le1\}} E\,(\,[\,e^{r\,T_n} + \pi\,(\Gamma_{n+1} - e^{r\,T_n})\,]^{\gamma}\,). \end{aligned} \tag{3.8}$$

Remark 3.6. Equation (3.4) and $C \ge 0$ imply that $W_{\tau_n} - b \ge 0$, and hence $X_{\tau_n} \ge Y_n$.

Proof of Theorem : The idea of the proof is similar to those in Hakansson(1970). Let the right-hand side of (2.14) be denoted by $S(X_{\tau_n})$ upon inserting (3.3) for $U(X_{\tau_n})$. Then we have

$$\begin{aligned} S(X_{\tau_n}) &= \max_{\{W_{\tau_n}, V_{\tau_n}\}} \{\, Q_n^{\nu}\, \frac{1}{\gamma}\,(W_{\tau_n} - b)^{\gamma} \\ &+\; Q_{n+1}^{\nu}\, A_{n+1}^{-\nu}\, e^{-\delta\,T_n}\, E\,[\,\frac{1}{\gamma}\,(X_{\tau_{n+1}} - Y_{n+1})^{\gamma} \mid \mathcal{H}_{\tau_n}]\,\}. \end{aligned}$$

Furthermore, let Y_n be defined by recurrence relationship

$$Y_n = b + (1-\varepsilon)^{-1}\, e^{-r\,T_n}\, Y_{n+1}, \tag{3.9}$$

with b, ε, r are as before. Then the total wealth process $X_{\tau_{n+1}}$ of (2.15) may be written as

$$\begin{aligned} X_{\tau_{n+1}} &= (1-\varepsilon)\,[\,X_{\tau_n} - Y_n - (W_{\tau_n} - b)\,]\, e^{r\,T_n} \\ &+\; V_{\tau_n}\,(\,\Gamma_{n+1} - e^{r\,T_n}\,) + Y_{n+1}. \end{aligned} \tag{3.10}$$

This implies that $S(X_{\tau_n})$ may be written as

$$\begin{aligned} S(X_{\tau_n}) &= \max_{\{W_{\tau_n}, V_{\tau_n}\}} \{Q_n^{\nu}\, \frac{1}{\gamma}\,(W_{\tau_n} - b)^{\gamma} + A_{n+1}^{-\nu}\, Q_{n+1}^{\nu}\, e^{-\delta T_n} \\ &\times\; E\,[\,\frac{1}{\gamma}\,(\,(1-\varepsilon)\,[\,X_{\tau_n} - Y_n - (W_{\tau_n} - b)\,]\, e^{r\,T_n} \\ &+\; V_{\tau_n}\,(\Gamma_{n+1} - e^{r\,T_n})\,)^{\gamma}\,]\,\}, \end{aligned} \tag{3.11}$$

subject to :

$$W_{\tau_n} - b \;\ge\; 0, \tag{3.12}$$

$$P\,\{\,X_{\tau_{n+1}} - Y_{n+1} \ge 0\,\} \;=\; 1, \tag{3.13}$$

$$V_{\tau_n} \;\ge\; 0. \tag{3.14}$$

To prevent the problem being trivial, the following assumption is imposed :

$$P\,\{\,\theta\,(\,\Gamma_{n+1} - e^{r\,T_n}\,) < 0\,\} > 0, \quad \text{for some } \ \theta > 0. \tag{3.15}$$

Notice that, for $X_{\tau_n} - Y_n - (W_{\tau_n} - b) > 0$, by re-arrangement, the total wealth process $X_{\tau_{n+1}}$ of (3.10) may be re-written as

$$\begin{aligned} X_{\tau_{n+1}} &= (1-\varepsilon)\,[X_{\tau_n} - Y_n - (W_{\tau_n} - b)] \\ &\times \ [e^{r\,T_n} + I_n\,(\Gamma_{n+1} - e^{r\,T_n})] + Y_{n+1}, \end{aligned} \tag{3.16}$$

with I_n is given by

$$I_n = \frac{V_{\tau_n}}{(1-\varepsilon)\,[\,X_{\tau_n} - Y_n - (W_{\tau_n} - b)\,]}.$$

This implies that $P\,\{\,X_{\tau_{n+1}} - Y_{n+1} \geq 0\,\} \;=\; 1,$ can only occur when either

$$X_{\tau_n} - Y_n - (W_{\tau_n} - b) = 0, \quad \text{and} \quad V_{\tau_n} = 0, \tag{3.17}$$

or,

$$X_{\tau_n} - Y_n - (W_{\tau_n} - b) > 0, \tag{3.18}$$

and

$$P\{e^{r\,T_n} + I_n\,(\Gamma_{n+1} - e^{r\,T_n}) \geq 0\} = 1. \tag{3.19}$$

Under feasibility with respect to (3.13), then

$$S(X_{\tau_n}) = \max\,\left\{\,Q_n^{\nu}\,\frac{1}{\gamma}\,(W_{\tau_n} - b)^{\gamma}\ ,\ \ \overline{S}(X_{\tau_n})\,\right\}, \tag{3.20}$$

where

$$\begin{aligned} \overline{S}(X_{\tau_n}) &= \sup_{\{\,W_{\tau_n},\,V_{\tau_n}\,\}}\{\,Q_n^{\nu}\,\frac{1}{\gamma}\,[\,W_{\tau_n} - b\,]^{\gamma} \\ &+ \ Q_{n+1}^{\nu}\,A_{n+1}^{-\nu}\,(1-\varepsilon)^{\gamma}\,e^{-\delta\,T_n}\,[\,X_{\tau_n} - Y_n - (W_{\tau_n} - b\,)\,]^{\gamma} \\ &\times \ E\,[\,\frac{1}{\gamma}\,(e^{rT_n} + I_n\,(\,\Gamma_{n+1} - e^{r\,T_n}\,))^{\gamma}]\}, \end{aligned} \tag{3.21}$$

subject to equations (3.12), (3.18), (3.19) and $I_n \geq 0$, since this is equivalent to equation (3.14) in view of (3.19).

The expectation factor in equation (3.21) may be re-written as $f(I_n)$, where f is defined by

$$f(\pi) = E\,[\,u(\exp(r\,T_n) + \pi(\Gamma_{n+1} - \exp(r\,T_n)))\,],$$

where utility function u is given by $u(C) = \frac{1}{\gamma}\,C^{\gamma}, \quad \gamma \in (0,1).$ Therefore

$$\begin{aligned} \overline{S}(X_{\tau_n}) &= \sup_{\{\,W_{\tau_n},\,V_{\tau_n}\,\}}\{\,Q_n^{\nu}\,\frac{1}{\gamma}\,[\,W_{\tau_n} - b\,]^{\gamma} + Q_{n+1}^{\nu}\ A_{n+1}^{-\nu} \\ &\times \ (1-\varepsilon)^{\gamma}\,e^{-\delta\,T_n}\,[\,X_{\tau_n} - Y_n - (W_{\tau_n} - b\,)\,]^{\gamma}\,f(I_n)\}. \end{aligned}$$

According to equation (3.8) and Lemma 3.1, the maximum of $f(I_n)$, subject to $I_n \geq 0$ and equation (3.19) is given by $\frac{1}{\gamma}\,\Omega_n$, where Ω_n is as in (3.8).

Then, by Lemma 3.1 results in $I_n = \pi_n$. Therefore,

$$V_{\tau_n} = (1-\varepsilon)\,[\,X_{\tau_n} - Y_n - (W_{\tau_n} - b)]\,\pi_n, \tag{3.22}$$

is optimal and unique for every W_{τ_n} which satisfies equations (3.12) and (3.18) when equation (3.19) holds. It can be shown that it is also optimal when equation (3.17) holds.

Since the second term of $\overline{S}(X_{\tau_n})$ is always nonnegative, then $\overline{S}(X_{\tau_n}) \geq Q_n^{\nu}\,\frac{1}{\gamma}\,[\,W_{\tau_n} - b\,]^{\gamma}$.
Therefore, equation (3.20) reduces to

$$S(X_{\tau_n}) = \max_{\{W_{\tau_n}\}} S^{W_{\tau_n}}(X_{\tau_n}), \tag{3.23}$$

where

$$S^{W_{\tau_n}}(X_{\tau_n}) = Q_n^{\nu}\,\frac{1}{\gamma}\,(W_{\tau_n} - b)^{\gamma} + Q_{n+1}^{\nu}\,A_{n+1}^{-\nu}\,R_n\,\frac{1}{\gamma}\,[X_{\tau_n} - Y_n - (W_{\tau_n} - b)]^{\gamma},$$

with $R_n = (1-\varepsilon)^{\gamma}\,e^{-\delta\,T_n}\,\Omega_n$. Since $u(C) = \frac{1}{\gamma}\,C^{\gamma}$ is strictly concave and $u'(0) = \infty$, then $S^{W_{\tau_n}}$ is strictly concave and differentiable with respect to W_{τ_n}, with a unique solution W_{τ_n} whenever $X_{\tau_n} - Y_n \geq 0$.

Differentiation of $S^{W_{\tau_n}}$ with respect to W_{τ_n} results in

$$\frac{dS^{W_{\tau_n}}}{dW_{\tau_n}} = Q_n^{\nu}\,(W_{\tau_n} - b)^{-\nu} - A_{n+1}^{-\nu}\,Q_{n+1}^{\nu}\,R_n\,[X_{\tau_n} - Y_n - (W_{\tau_n} - b)]^{-\nu}.$$

By setting $dS^{W_{\tau_n}}/dW_{\tau_n} = 0$, then we have

$$(W_{\tau_n} - b)^{-\nu} = A_{n+1}^{-\nu}\,Q_n^{-\nu}\,Q_{n+1}^{\nu}\,R_n\,[\,X_{\tau_n} - Y_n - (W_{\tau_n} - b)\,]^{-\nu},$$

from which results in

$$(W_{\tau_n} - b)\,[1 + A_{n+1}\,Q_n\,Q_{n+1}^{-1}\,R_n^{-1/\nu}\,] = A_{n+1}\,Q_n\,Q_{n+1}^{-1}\,R_n^{-1/\nu}\;(X_{\tau_n} - Y_n).$$

Let define F_n as the following :

$$F_n = \frac{A_{n+1}\,Q_n}{A_{n+1}\,Q_n + Q_{n+1}\,R_n^{1/\nu}}.$$

Then the optimal withdrawal processes W_{τ_n} may be written as

$$W_{\tau_n} = F_n\,(X_{\tau_n} - Y_n) + b. \tag{3.24}$$

By insertion of relation (3.24) into equation (3.22) results in the optimal investment strategy processes V_{τ_n} is in the form

$$V_{\tau_n} = (1-\varepsilon)\,(1 - F_n)\,(X_{\tau_n} - Y_n)\,\pi_n. \tag{3.25}$$

By substitution of (3.24) into (3.23) yields

$$\begin{aligned} S(X_{\tau_n}) &= Q_n^\nu \, F_n^\gamma \, \frac{1}{\gamma} \, (X_{\tau_n} - Y_n)^\gamma + Q_{n+1}^\nu \, A_{n+1}^{-\nu} \, R_n \\ &\times \frac{1}{\gamma} \, (X_{\tau_n} - Y_n)^\gamma \, (1 - F_n)^\gamma \\ &= Q_n^\nu \, A_n^{-\nu} \, \frac{1}{\gamma} \, (X_{\tau_n} - Y_n)^\gamma \, [\, A_n^\nu \, F_n^\gamma \\ &+ A_n^\nu \, A_{n+1}^{-\nu} \, Q_n^{-\nu} \, Q_{n+1}^\nu \, R_n \, (1 - F_n)^\gamma \,]. \end{aligned}$$

On the other hand, from relation (3.3), $U(X_{\tau_n})$ is given by

$$U(X_{\tau_n}) = Q_n^\nu \, A_n^{-\nu} \, \frac{1}{\gamma} \, (X_{\tau_n} - Y_n)^\gamma.$$

This implies that $S(X_{\tau_n}) = U(X_{\tau_n})$ if and only if

$$A_n^\nu \, F_n^\gamma + A_n^\nu \, A_{n+1}^{-\nu} \, Q_n^{-\nu} \, Q_{n+1}^\nu \, R_n \, (1 - F_n)^\gamma = 1. \quad (*)$$

But relation $(*)$ holds if and only if

$$\begin{aligned} [A_{n+1} \, Q_n + Q_{n+1} \, R_n^{1/\nu}]^\gamma &= A_n^\nu \, A_{n+1}^{-\nu} \, Q_n^{-\nu} + A_n^\nu \, A_{n+1}^{-\nu} \, Q_n^{-\nu} \, Q_{n+1} \, R_n^{1/\nu} \\ &= A_n^\nu \, A_{n+1}^{-\nu} \, Q_n^{-\nu} \, [A_{n+1} \, Q_n + Q_{n+1} \, R_n^{1/\nu}]. \end{aligned}$$

Therefore, $(*)$ holds if

$$A_n = \frac{A_{n+1} \, Q_n}{A_{n+1} \, Q_n + Q_{n+1} \, R_n^{1/\nu}} = F_n.$$

Hence, the proof of the Theorem has been completed ■

Remark 3.7. For an equal intervals problem, that is $T_n = T_{n+1} \quad \forall n$, then

$$A_n = 1 - R_n^{1/\nu}, \quad \text{and} \quad Y_n = \frac{(1 - \epsilon) b}{(1 - \epsilon) - e^{-r T_n}}.$$

Corollary 2. Let the control problem be defined by problem (2.14)-(2.15). Furthermore, let T_n be fixed for $n = 1, 2, 3, ...$ Then A_n as given by (3.6) has a property such that either $A_n \geq Q_n \, (1 - R_n^{1/\nu})$, or $A_n < Q_n \, (1 - R_n^{1/\nu})$.

Proof : By Remark 3.7, for a given set of $\{T_n\}$ then $\{A_n\}$ will satisfy either $A_n \geq 1 - R_n^{1/\nu}$ or, $A_n < 1 - R_n^{1/\nu}$. By keeping in mind that $0 \leq Q_n \leq 1$, then $\{A_n\}$ will also satisfy either $A_n \geq Q_n (1 - R_n^{1/\nu})$ or $A_n < Q_n (1 - R_n^{1/\nu})$. ■

Remark 3.8. Suppose that A_{n+1} satisfies recurrence relationship (3.6). If A_n as given by relation (3.6) has a property such that $A_n \geq Q_n \, (1 - R_n^{1/\nu})$, then A_{n+1} is such that

$$\frac{A_{n+1}}{Q_{n+1}} \geq (1 - R_n^{1/\nu}) \, R_n^{1/\nu}.$$

Proof : By applying $A_n \geq Q_n\ (1 - R_n^{1/\nu})$, in relation (3.6) and arranging the terms, then A_{n+1} satisfies

$$\begin{aligned}\frac{A_{n+1}}{Q_{n+1}} &= \frac{A_n\ R_n^{1/\nu}}{Q_n\ (1 - A_n)} \\ &\geq \frac{Q_n\ (1 - R_n^{1/\nu})\ R_n^{1/\nu}}{Q_n\ (1 - A_n)} \\ &\geq \frac{(1 - R_n^{1/\nu})\ R_n^{1/\nu}}{[1 - Q_n\ (1 - R_n^{1/\nu})]}.\end{aligned}$$

By its definition, $Q_n = 1 - e^{-\delta/\nu T_n} \in [0, 1]$. Meanwhile, according to Lemma 3.4, $R_n \leq (1 - \varepsilon)^\gamma$. Since $\varepsilon < 1$ and $0 < \gamma < 1$, then $R_n \in [0, 1]$. Therefore, $1 - Q_n\ (1 - R_n^{1/\nu}) \leq 1$. Hence,

$$\frac{A_{n+1}}{Q_{n+1}} \geq (1 - R_n^{1/\nu})\ R_n^{1/\nu} \quad \blacksquare$$

4. Conclusion

We have established an investment strategy for a consumption and investment selection problem for an individual who seeks to maximize the expected utility of consumption. The individual has available a riskless asset with fixed interest rate and a risky one with logarithmic Brownian motion price fluctuations. The individual observes current wealth when making transaction, and decisions to transact can be made at any time, but not without costs. The individual is charged a fixed fraction $\epsilon > 0$ of the current wealth as a portfolio management fee plus fixed transaction costs. The problem was formulated in discrete-continuous-time stochastic optimal control problem.

A solution to the consumption and investment strategy selection for the individual was derived. The first conjecture in discrete-continuous-time setting is that it is not optimal for an individual to take more money out of his/her portfolio than that it is needed for consumption during intervals.

For given transaction intervals, we derived the optimal value function as well as the optimal strategy for the withdrawal process and the investment in the risky asset. We showed that, for each interval, the optimal value function as well as the optimal withdrawal process and the investment strategy in the risky asset were obtained by dynamic programming.

References

[1] Constantinides, G.M., 1979, Multiperiod Consumption and Investment Behaviour with Transaction Costs, *Management Science*, 25(11), pp. 1127-1137

[2] Constantinides, G.M., 1986, Capital Market Equilibrium with Transaction Costs, *Journal of Political Economy*, 94(4), pp. 842-862

[3] Cox, J.C and C.F Huang, 1989, Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process, *Journal of Economic Theory*, 49, pp. 33-83

[4] Davis, M.H.A and A.R Norman, 1990, Portfolio Selection with Transaction Costs, *Mathematics of Operations Research*, 15(4), pp. 676-713

[5] Duffie, D and T.S Sun, 1990, Transaction Costs and Portfolio Choice in a Discrete-Continuous-Time Setting, *Journal of Economic Dynamics and Control*, 14, pp. 35-51

[6] Hakansson, N.H., 1970, Optimal Investment and Consumption Strategies Under Risks for a Class of Utility Functions, *Econometrica*, 38(5), pp.587-607

[7] Karatzas, I. and S.E. Shreve, 1988, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York

[8] Magill M.J.P and G.M. Constantinides, 1976, Portfolio Selection with Transaction Costs, *Journal of Economic Theory*, 13, pp. 245-263

[9] Merton, R.C., 1990, *Continuous-Time Finance*, Blackwell, Cambridge

[10] Merton, R.C., 1971, Optimum Consumption and Portfolio Rules in Continuous Time Model, *Journal of Economic Theory*, 3, pp. 373-413

[11] Protter, P., 1990, *Stochastic Intergration and Differential Equations*, Springer-Verlag, Berlin

[12] Syahril, E., 2003, An Optimal Control Formulation of Portfolio Selection Problem with Bullet Transaction Cost, *Jurnal Matematika dan Aplikasinya*, Vol 2, No 1, pp. 25-36